THE CHOW RING OF THE MODULI SPACE AND ITS RELATED HOMOGENEOUS SPACE OF BUNDLES ON $\mathbb{P}^2$ WITH CHARGE 1

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ABSTRACT. For an algebraically closed field K with $\operatorname{ch}(K) \neq 2$, let $\mathcal{O}M(1, SO(n, K))$ denote the moduli space of holomorphic bundles on $\mathbb{P}^2$ with the structure group $SO(n, K)$ and half the first Pontryagin index being equal to 1, each of which is trivial on a fixed line l_∞ and has a fixed holomorphic trivialization there. In this paper we determine the Chow ring of $\mathcal{O}M(1, SO(n, K))$.

1. INTRODUCTION

Let G be one of the classical groups $SU(n)$, $SO(n)$ or $Sp(n)$, and let $k \geq 0$ be half the first Pontryagin index of a G-bundle P over $S^4 = \mathbb{R}^4 \cup \{\infty\}$. Denote by $M(k, G)$ the framed moduli space whose points represent isomorphism classes of pairs:

$$\text{(self-dual } G\text{-connections on } P, \text{ isomorphism } P_\infty \simeq G).$$

Let $\mathcal{O}M(k, G^\mathbb{C})$ denote the moduli space of holomorphic bundles on $\mathbb{CP}^2$ for the associated complex group, trivial on a fixed line l_∞ and with a fixed holomorphic trivialization there. Then Donaldson ([7]) showed a diffeomorphism $M(k, G) \simeq \mathcal{O}M(k, G^\mathbb{C})$.

In [12] the topology of $M(1, SO(n)) \simeq \mathcal{O}M(1, SO(n, \mathbb{C}))$ was studied in detail. The result was used in [11] to prove the fact that the natural homomorphism $J : H_.(M(1, SO(n)), \mathbb{Z}/2) \to H_.(\Omega_0^3 \mathrm{Spin}(n), \mathbb{Z}/2)$ is injective. Moreover, the image of J was determined. To prove this, the following description of $\mathcal{O}M(1, SO(n, \mathbb{C}))$ by a homogeneous space was used: We set

$$W_n = SO(n)/(SO(n-4) \times SU(2)).$$

Then there is a diffeomorphism

$$(1.1) \qquad \mathcal{O}M(1, SO(n, \mathbb{C})) \simeq \mathbb{R}^5 \times W_n.$$

The purpose of this paper is to generalize the definition of $\mathcal{O}M(1, SO(n, \mathbb{C}))$ for any algebraically closed field K with $\mathrm{ch}(K) \neq 2$ and to determine the Chow ring of this. The Chow ring of a classifying space was studied by Totaro [19]. A loop space is considered to be a dual situation of a classifying space in a certain sense. Our result and the result of [11] are the first step for a loop space.

Definition 1.1. Let K be an algebraically closed field with $\mathrm{ch}(K) \neq 2$. Let $\mathcal{O}M(1, SO(n, K))$ denote the moduli space of holomorphic bundles on $\mathbb{P}^2$ with the structure group $SO(n, K)$ and half the first Pontryagin index being equal to 1, each of which is trivial on a fixed line l_∞ and has a fixed holomorphic trivialization there.

The moduli space $\mathcal{O}M(1, SO(n, K))$ is a quasi-projective variety and defines the Chow ring. More explicitly, the diffeomorphism (1.1) is generalized (in the sense of a biregular map) as follows: We set

$$X_n = SO(n, K)/(SO(n - 4, K) \times SL(2, K)) \cdot P_u,$$

where P_u denotes the unipotent radical. (Recall that for a parabolic subgroup P of an algebraic group G, P is a semidirect product of a reductive group and its unipotent radical P_u.) Then there is a biregular map

$$(1.2) \qquad \mathcal{O}M(1, SO(n, K)) \simeq \mathbb{A}^2 \times X_n.$$

(For the proof of (1.2), see Proposition 2.1.) A formula of Grothendieck [3] shows that

$$CH^\cdot(\mathcal{O}M(1, SO(n, K))) \simeq CH^\cdot(X_n).$$

The purpose of this paper is to determine the Chow rings of X_n and its related algebraic variety Y_n explicitly.

The Schubert cell approach of the Chow ring of Y_n by using a Young diagram is done in [15], [16]. However it needs further work to determine the Chow ring of X_n from this. Hence we first calculate the Chow ring of Y_n more explicitly by a different method. Then we calculate the Chow ring from the results. Our results for the Chow ring of X_n are new.

This paper is organized as follows. In Sect. 2 we first prove (1.2). Then we recall basic facts on the Chow ring. In Sect. 3 we determine an integral basis and the ring structure of $CH^\cdot(Y_n)$, where Y_n is an algebraic variety which is related to X_n. (See Theorems 3.7 and 3.9.) The ring structure of $CH^\cdot(Y_n)$ proved in Theorem 3.9 is one of our main results. Since the results are long, we give them in tables in Sect. 5. (See 5.2-5.5.) Using the results of Sect. 3, we determine $CH^\cdot(X_n)$ in Sect. 4. (See Theorem 4.1.)

We thank N. Yagita for turning our interest to the Chow ring and explaining the paper [17].

2. Preliminaries

We first prove (1.2):

Proposition 2.1. *For an algebraically closed field K with $\mathrm{ch}(K) \neq 2$, there is a biregular map*

$$\mathcal{O}M(1, SO(n, K)) \simeq \mathbb{A}^2 \times X_n.$$

Proof . Recall that a monad description of $\mathcal{O}M(k, SO(n, \mathbb{C}))$ was indicated in [7] and given explicitly in [14] and [18]. It is easy to see that the description remains valid for any algebraically closed field K. In particular, $\mathcal{O}M(1, SO(n, K))$ is given as follows:

Lemma 2.2. *Let $\mathscr{C}_n$ be the space of $n \times 2$ matrices*

$$c = \begin{pmatrix} z_1 & w_1 \\ z_2 & w_2 \\ \vdots & \vdots \\ z_n & w_n \end{pmatrix}$$

with coefficients in K satisfying:

   a) $c^T c = O$, *that is:*

$$\sum_{i=1}^{n} z_i^2 = 0, \quad \sum_{i=1}^{n} w_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{n} z_i w_i = 0,$$

   b) *The rank of c over K is 2.*

The group $SL(2, K)$ acts on $\mathscr{C}_n$ from the right by the multiplication of matrices. Then there is a biregular map

$$\mathcal{O}M(1, SO(n, K)) \simeq \mathbb{A}^2 \times (\mathscr{C}_n / SL(2, K)).$$

From the lemma, it suffices to prove $X_n \simeq \mathscr{C}_n / SL(2, K)$. We prove this for the case $n = 2m$. (The case $n = 2m + 1$ can be proved similarly.) Recall that in [2], $SO(n, K)$ was defined as follows: Let $q(x)$ be a quadratic form on $\mathbb{A}^n$ defined by $q(x) = \sum_{i=1}^{m} x_i x_{m+i}$, and let $B(x, y)$ be the associated bilinear form. Then $SO(n, K)$ is defined by

$$SO(n, K) = \{\sigma \in \mathrm{Aut}(\mathbb{A}^n) : B(\sigma(x), \sigma(y)) = B(x, y) \text{ for } x, y \in \mathbb{A}^n\}.$$

We set

$$x_j = z_j + \sqrt{-1} z_j, \ x_{m+j} = z_j - \sqrt{-1} z_j, \ y_j = w_j + \sqrt{-1} w_j \quad \text{and} \quad y_{m+j} = w_j - \sqrt{-1} w_j,$$

where $1 \leq j \leq m$. Then the defining equations of $\mathscr{C}_n$ are given by

$$q(x) = q(y) = 0 \quad \text{and} \quad B(x, y) = 0.$$

Clearly $SO(n, K)$ acts on $\mathscr{C}_n$. It is easy to prove the following lemma. (See [2, V 23.4].)

Lemma 2.3.

$$SO(n, K)/SO(n - 4, K) \cdot P_u \simeq \mathscr{C}_n,$$

where P_u is the unipotent radical of a parabolic subgroup with a Levi factor $SO(n - 4, K) \times GL(2, K)$.

Now Proposition 2.1 follows from Lemma 2.3. This completes the proof of Proposition 2.1. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next we recall basic facts on the Chow ring. We suppose that an algebraic variety V is defined over K. Let $CH^{\cdot}(V)$ denote the Chow ring and $CH^i(V)$ the subgroup of $CH^{\cdot}(V)$ generated by the cycles of codimension i.

Theorem 2.4 ([3])**.** (i) *Let V be a nonsingular variety, X a nonsingular closed subvariety of V, and $U = X - V$. Then there exists an exact sequence*

$$CH^{\cdot}(X) \xrightarrow{i_*} CH^{\cdot}(V) \xrightarrow{j^*} CH^{\cdot}(U) \to 0,$$

where $i : X \to V$ (resp. $j : U \to V$) is a closed immersion (resp. an open immersion).
  For the definitions of i_* and j^*, see also [10].
  (ii) *Let $\pi : E \to V$ be a fiber bundle with an affine space $\mathbb{A}^n$ as a fiber. Then the induced map $\pi^* : CH^{\cdot}(V) \to CH^{\cdot}(E)$ is an isomorphism.*

The Chow ring of the following projective variety is well-known.

Theorem 2.5 ([1], [6])**.** *Let G be a reductive algebraic group and P a maximal parabolic subgroup. Then*
  (i) *a quotient G/P is a nonsingular projective variety.*
  (ii) *$CH^{\cdot}(G/P)$ is generated by the Schubert varieties.*
  (iii) *$CH^{\cdot}(G/P)$ is independent of $\mathrm{ch}(K)$. Moreover, $CH^{\cdot}(G/P) \simeq H^{\cdot}(G/P, \mathbb{Z})$ for $K = \mathbb{C}$.*

3. The ring structure of $CH^{\cdot}(Y_n)$

Before describing the results, we need some notations and results. We set

$$Y_n = SO(n, K)/(SO(n - 4, K) \times GL(2, K)) \cdot P_u.$$

Then we have a principal bundle

$$(3.1) \qquad \mathbb{G}_m \to X_n \xrightarrow{\pi} Y_n.$$

In this section we determine an integral basis and the ring structure of $CH^{\cdot}(Y_n)$. By Theorem 2.5 (ii), (iii), we obtain the following theorem:

Theorem 3.1 ([12]). *We have an isomorphism as modules*:
  (1) *For $n = 2m$,*

$$CH^{\cdot}(Y_n) \otimes \mathbb{Z}/2 \simeq \mathbb{Z}/2[c_1, c_2]/(b_{m-1}, c_2 b_{m-2}) \otimes \Delta(v_{2m-4}, v_{2m-2}).$$

  (2) *For $n = 2m + 1$,*

$$CH^{\cdot}(Y_n) \otimes \mathbb{Z}/2 \simeq \mathbb{Z}/2[c_1, c_2]/(b_{m-1}, c_2 b_{m-2}) \otimes \Delta(v_{2m-2}, v_{2m}),$$

where $|c_1| = 2, |c_2| = 4, |b_i| = 2i$ and $|v_i| = i$.

Theorem 3.2 ([12]). *Let p be an odd prime. Then we have a ring isomorphism*:
  (1) *For $n = 2m$,*

$$CH^{\cdot}(Y_n) \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p[c_1, c_2, \chi_{2m-4}]/(c_2 \chi_{2m-4}, \chi_{2m-4}^2 - d_{m-2}, d_{m-1}),$$

where $\chi_{2m-4} \in H^{2m-4}(BSO_{2m-4}, \mathbb{Z}/p)$ is the Euler class.
  (2) *For $n = 2m + 1$,*

$$CH^{\cdot}(Y_n) \otimes \mathbb{Z}/p \simeq \mathbb{Z}/p[c_1, c_2]/(d_{m-1}, c_2^2 d_{m-2}).$$

We recall the definitions of b_i, d_i and v_i. In a polynomial ring $\mathbb{Z}[\alpha, \beta]$, we set $c_1 = \alpha + \beta$ and $c_2 = \alpha\beta$. Then b_k and d_k are defined by

$$b_k = (-1)^k \sum_{i=0}^{k} \alpha^i \beta^{k-i}$$

and

$$d_k = (-1)^k \sum_{i=0}^{k} \alpha^{2i} \beta^{2k-2i}.$$

The element $v_{2r} \in CH^{2r}(Y_n)$ is defined by
  (1) For $n = 2m$,

$$(3.2) \qquad \begin{cases} 2v_{2m-4} = \chi_{2m-4} - b_{m-2} \\ 2v_{2m-2} = b_{m-1}. \end{cases}$$

  (2) For $n = 2m + 1$,

$$(3.3) \qquad \begin{cases} 2v_{2m-2} = b_{m-1} \\ 2v_{2m} = c_2 b_{m-2}. \end{cases}$$

The following formulas are easily proved.

Lemma 3.3. *We have*

$$b_k = (-1)^k \sum_{\mu=0}^{\left[\frac{k}{2}\right]} (-1)^\mu \binom{k-\mu}{\mu} c_1^{k-2\mu} c_2^\mu$$

and

$$d_k = (-1)^k \sum_{\mu=0}^{k} (-1)^\mu \binom{2k-\mu+1}{\mu} c_1^{2k-2\mu} c_2^\mu.$$

The following lemmas are also easily shown.

Lemma 3.4.

$$\sum_{\mu=0}^{h} (-1)^\mu c_2^{h-\mu} b_{2\mu} = d_h.$$

Lemma 3.5. *We set $f_n(x) = (1+x)^n - (1+x^n)$ and write $f_n(x)$ as*

$$f_n(x) = \sum_{\mu=1}^{\left[\frac{n}{2}\right]} a_\mu x^\mu (1+x)^{n-2\mu}.$$

Then we have

$$a_\mu = (-1)^{\mu+1} \frac{n}{\mu} \binom{n-1-\mu}{\mu-1}.$$

Especially, the last term is given by

$$\begin{cases} (-1)^{s+1} 2x^s & \text{for } n = 2s \\ (-1)^{s+1}(2s+1)x^s(1+x) & \text{for } n = 2s+1. \end{cases}$$

For $n = 2m$ or $2m+1$, we define a subgroup A_n of $CH^\cdot(Y_n)$ by

$$(3.4) \qquad A_n = \left(\bigoplus_{i=0}^{m-2} \mathbb{Z}[c_1]/(c_1^{m-1-i})\{c_2^i\} \right) \otimes B_n,$$

where we set

$$B_n = \begin{cases} \Delta_\mathbb{Z}(v_{2m-4}, v_{2m-2}) & n = 2m \\ \Delta_\mathbb{Z}(v_{2m-2}, v_{2m}) & n = 2m+1. \end{cases}$$

The generators v_{2i} is specified in (3.2) and (3.3).

The following lemma is proved in the same way as in [12, Lemma 3.8].

Lemma 3.6. *For a prime p, we abbreviate $CH^{\cdot}(Y_n) \otimes \mathbb{Z}_{(p)}$ as $CH^{\cdot}(Y_n)_{(p)}$.
If p is odd, we have the following isomorphism of modules:*
    (i) *For $n = 2m$,*

$$CH^{\cdot}(Y_n)_{(p)} \simeq \mathbb{Z}_{(p)}[c_1]/(c_1^{2(m-1)})\{1, \chi_{2m-4}\} \oplus \overset{m-2}{\underset{i=1}{\oplus}} \mathbb{Z}_{(p)}[c_1]/(c_1^{2(m-1-i)})\{c_2^{2i-1}, c_2^{2i}\}.$$

    (ii) *For $n = 2m + 1$,*

$$CH^{\cdot}(Y_n)_{(p)} \simeq \overset{m-2}{\underset{i=0}{\oplus}} \mathbb{Z}_{(p)}[c_1]/(c_1^{2(m-1-i)})\{c_2^{2i}, c_2^{2i+1}\}.$$

Theorem 3.7. *An integral basis of $CH^{\cdot}(Y_n)$ is constructed from the
monomial basis of A_n in (3.4). The results are summed up in Sect. 5,
5.2.*

The proof is based on a rather complicated calculation. Its outline is
as follows: We construct a set of suitable generators starting from the
basis of A_n. It is easily verified that it is a basis of $CH^{\cdot}(Y_n) \otimes \mathbb{Z}/p$ by
using the presentation of Lemma 3.6. Then it is a $\mathbb{Z}$-basis of $CH^{\cdot}(Y_n)$.
We only prove the case $n = 2m + 1$ and m is even. To simplify the
proof, we set:

(3.5)

$C = CH^{\cdot}(Y_n)$

$$\mathfrak{b}_{m-1} = (-1)^{m-1} b_{m-1} = \sum_{i+j=m-1} \alpha^i \beta^j, \quad e_m = c_2 \mathfrak{b}_{m-2}$$

$\mathfrak{d}_{m-1} = (-1)^{m-1} d_{m-1}$

$\mathcal{A} = \left(\mathbb{Z}[c_1]/(c_1^{m-1}) \oplus \cdots \oplus \mathbb{Z}[c_1]/(c_1^{m-1-i})\{c_2^i\} \oplus \cdots \oplus \{c_2^{m-2}\} \right) \otimes \mathbb{Z}\{1, \mathfrak{b}_{m-1}, e_m, \mathfrak{b}_{m-1}e_m\}.$

Let $\mathcal{A}_n$ the set of a basis

$$\{c_1^i c_2^j \mathfrak{b}_{m-1}^{\epsilon_1} e_m^{\epsilon_2} : i + j \leq m - 2, \epsilon_k = 0 \text{ or } 1 \ (k = 1, 2)\}.$$

Then we have

$$C \subset \mathbb{Q}[c_1, c_2]/(\mathfrak{d}_{m-1}, c_2^2 \mathfrak{d}_{m-2}), \quad A \subset \mathbb{Q}[c_1, c_2]/(\mathfrak{d}_{m-1}, c_2^2 \mathfrak{d}_{m-2})$$

$$\text{and} \quad c_2^{2i} \mathfrak{d}_{m-1-i} \in (\mathfrak{d}_{m-1}, c_2^2 \mathfrak{d}_{m-2}), \ i \geq 0.$$

Our concern is a homogeneous polynomial algebra $S_{\mathbb{Z}}(V)$ of $V = \{c_1, c_2\}$. It is identified with an inhomogeneous ring $\mathbb{Z}[x]$ by putting
$x = \frac{\beta}{\alpha}$. Then we have

$$c_1 = 1 + x, \ c_2 = x, \ \mathfrak{b}_{m-1} = \frac{x^m - 1}{x - 1}, \quad \text{and} \quad \mathfrak{d}_{m-1} = \frac{x^{2m} - 1}{x^2 - 1}.$$

Using the identification, the next formulas are directly checked: In $\mathbb{Z}[c_1, c_2]$,

$$- c_2^{2i+1+j}\mathfrak{b}_{m-2-2i} = c_2^{j+1}\mathfrak{b}_{2i-1}\mathfrak{b}_{m-1} - c_2^j\mathfrak{b}_{2i}e_m$$

and

$$- c_2^{2i+2+j}\mathfrak{b}_{m-3-2i} = c_2^{j+1}\mathfrak{b}_{2i}\mathfrak{b}_{m-1} - c_2^j\mathfrak{b}_{2i+1}e_m.$$

We set

$$(3.6) \quad \left[c_1^{2i-1}c_2^{j+1}\mathfrak{b}_{m-1}\right] := c_2^{j+1}\mathfrak{b}_{2i-1}\mathfrak{b}_{m-1} - c_2^j\mathfrak{b}_{2i}e_m = -c_2^{2i+j+1}\mathfrak{b}_{m-2-2i}$$

and

$$\left[c_1^{2i}c_2^{j+1}\mathfrak{b}_{m-1}\right] := c_2^{j+1}\mathfrak{b}_{2i}\mathfrak{b}_{m-1} - c_2^j\mathfrak{b}_{2i+1}e_m = -c_2^{2i+j+2}\mathfrak{b}_{m-3-2i}.$$

Noting $c_2^2\mathfrak{d}_{m-2} = 0$, we have

$$c_2^2\mathfrak{b}_{m-2}\mathfrak{b}_{m-1} - c_2^2c_1\mathfrak{d}_{m-2} = \frac{x^3}{(x-1)^2}(x^{m-2}-1)(x^{m-1}-1) = c_2^3\mathfrak{b}_{m-3}\mathfrak{b}_{m-2}.$$

Using $c_2^{2i}\mathfrak{d}_{m-1-i} = 0$, $1 \le i \le m$, we repeat the argument and get

$$(3.7) \qquad\qquad c_2^i\mathfrak{b}_{m-1}e_m = c_2^{2i+1}\mathfrak{b}_{m-2-i}\mathfrak{b}_{m-1-i}.$$

In Y_n, we see that $\mathfrak{d}_{m-1} = 0$. Recall that we set $f_{2i+3}(x) = (1+x)^{2i+3} - (1+x^{2i+3})$ (see Lemma 3.5). Then we have

$$c_1^{2i+1}c_2^j\mathfrak{b}_{m-1}e_m - c_2^{j+1}\mathfrak{d}_i\mathfrak{d}_{m-1}$$

$$= \frac{x^{j+1}(x^m-1)}{(x^2-1)^2}\left((1+x)^{2i+3}(x^{m-1}-1) - (x^{2i+2}-1)(x^m+1)\right)$$

$$= \frac{x^{j+1}(x^m-1)}{(x^2-1)^2}\left((1+x^{2i+3})(x^{m-1}-1) - (x^{2i+2}-1)(x^m+1) + f_{2i+3}(x)(x^{m-1}-1)\right)$$

$$= \frac{x^{2i+j+3}}{(x-1)(x^2-1)}(x^m-1)(x^{m-2i-3}-1) + \frac{x^{j+1}}{(x^2-1)^2}(x^{m-1}-1)(x^m-1)f_{2i+3}(x).$$

We set

$$I_1 = \frac{x^{2i+j+3}}{(x-1)(x^2-1)}(x^m-1)(x^{m-2i-3}-1)$$

and

$$I_2 = \frac{x^{j+1}}{(x^2-1)^2}(x^{m-1}-1)(x^m-1)f_{2i+3}(x).$$

Since $x^{2\lambda}\mathfrak{d}_{m-1-\lambda} = 0$ for $0 \leq \lambda \leq m-1$, we see that $x^{2i+j+3+\lambda}\mathfrak{d}_{m-1-(i+2)-\lambda} = 0$ for $0 \leq \lambda \leq j-1$. Hence

$$I_1 - \sum_{\lambda=0}^{j-1} x^{2i+j+3+\lambda}\mathfrak{d}_{m-1-(i+2)-\lambda} = \frac{x^{2i+2j+3}}{(x-1)(x^2-1)}(x^{m-j}-1)(x^{m-2i-j-3}-1).$$

From Lemma 3.5,

$$I_2 = \sum_{\mu=1}^{i} a_\mu c_2^{j+\mu} c_1^{2i+1-2\mu}\mathfrak{b}_{m-1}e_m + (-1)^i k x^{i+j+2}\frac{(x^{m-1}-1)(x^m-1)}{(x-1)(x^2-1)},$$

where $k = 2i+3$.

We set $J = x^{i+j+2}\frac{(x^{m-1}-1)(x^m-1)}{(x-1)(x^2-1)}$. Using $x^{i+j+2+\lambda}\mathfrak{d}_{m-2-\lambda} = 0$ for $1 \leq \lambda \leq i+j$, we see that

$$J - \sum_{\lambda=0}^{i+j} x^{i+j+2+\lambda}\mathfrak{d}_{m-2-\lambda} = \frac{x^{2i+2j+3}}{(x-1)(x^2-1)}(x^{m-2-i-j}-1)(x^{m-1-i-j}-1).$$

Afterwards, we introduce the notation

$$(3.8) \quad \left[c_1^{2i+1}c_2^j\mathfrak{b}_{m-1}e_m\right] := c_1^{2i+1}c_2^j\mathfrak{b}_{m-1}e_m - \sum_{\mu=1}^{i} a_\mu c_2^{j+\mu}c_1^{2i+1-2\mu}\mathfrak{b}_{m-1}e_m,$$

where $f_{2i+3}(x) = \sum_{\mu=1}^{i+1} a_\mu x^\mu(1+x)^{2i+3-2\mu}$ (see Lemma 3.5).

We sum up these arguments:

$$(3.9) \qquad\qquad \left[c_1^{2i+1}c_2^j\mathfrak{b}_{m-1}e_m\right] = I_1 + (-1)^i kJ,$$

where

$$I_1 = \frac{x^{2i+2j+3}}{(x-1)(x^2-1)}(x^{m-j}-1)(x^{m-2i-j-3}-1),$$

$$J = \frac{x^{2i+2j+3}}{(x-1)(x^2-1)}(x^{m-2-i-j}-1)(x^{m-1-i-j}-1), \quad \text{and} \quad k = 2i+3.$$

A direct calculation shows $J - I_1 = \frac{x^{m+j}}{(x-1)(x^2-1)}(x^{i+1}-1)(x^{i+2}-1)$. Hence we get a key formula
(3.10)

$$\left[c_1^{2i+1}c_2^j\mathfrak{b}_{m-1}e_m\right] = \left((-1)^i(2i+3)+1\right)I_1 + (-1)^i(2i+3)\frac{x^{m+j}}{(x-1)(x^2-1)}(x^{i+1}-1)(x^{i+2}-1).$$

We note that (3.9) holds for m even or odd. From now on, we assume that m is even. When we put $j = 1$ in (3.9), we see that $I_1 = c_2^{2i+4}\mathfrak{d}_{\frac{m}{2}-i-3}e_m$. We set

$$\langle c_1^{2i+1}c_2\mathfrak{b}_{m-1}e_m\rangle := \left[c_1^{2i+1}c_2\mathfrak{b}_{m-1}e_m\right] - \left((-1)^i(2i+3)+1\right)c_2^{2i+4}\mathfrak{d}_{\frac{m}{2}-i-3}e_m.$$

Then we have

(3.11)
$$\frac{\langle c_1^{2i+1} c_2 \mathfrak{b}_{m-1} e_m \rangle}{2i+3} = (-1)^i c_2^{m+1} \frac{\mathfrak{b}_i \mathfrak{b}_{i+1}}{c_1}.$$

We call a generator $c_1^{m-3-2i} c_2^{2i+1} \mathfrak{b}_{m-1} e_m$ to be the head of the presentation $\mathcal{A}_n$ in (3.4). For the head generator, it is directly shown that $I_1 = 0$ by (3.9). Hence the formula (3.9) implies

(3.12)
$$\frac{\left[c_1^{m-3-2i} c_2^{2i+1} \mathfrak{b}_{m-1} e_m \right]}{m-1-2i} = (-1)^i \frac{\mathfrak{b}_{\frac{m}{2}-i-2} \mathfrak{b}_{\frac{m}{2}-i-1}}{c_1}.$$

Then the formulas (3.10) and (3.11) imply the following relations: We set

$$\langle c_1^{2i+1} \mathfrak{b}_{m-1} e_m \rangle := \left[c_1^{2i+1} \mathfrak{b}_{m-1} e_m \right] - \left((-1)^i (2i+3) + 1 \right) c_2^{2i+2} \mathfrak{d}_{\frac{m}{2}-i-2} e_m$$

and

$$\langle c_1^{2\alpha+1-2\beta} c_2^{1+\beta} \mathfrak{b}_{m-1} e_m \rangle := \left[c_1^{2\alpha+1-2\beta} c_2^{1+\beta} \mathfrak{b}_{m-1} e_m \right]$$
$$- \left((-1)^{\alpha-\beta} (2\alpha - 2\beta + 3) + 1 \right) c_2^{2\alpha+4} \mathfrak{d}_{\frac{m}{2}-\alpha-3} e_m.$$

Then we have
(3.13)
$$\langle c_1^{2i+1} \mathfrak{b}_{m-1} e_m \rangle - \frac{(-1)^i (2i+3) + 1}{(-1)^{i-1}(2i+1)} \langle c_1^{2i-1} c_2 \mathfrak{b}_{m-1} e_m \rangle = -\frac{c_2^m}{c_1} \mathfrak{b}_i \mathfrak{b}_{i+1}$$

(3.14)
$$\langle c_1^{2\alpha+1-2\beta} c_2^{1+\beta} \mathfrak{b}_{m-1} e_m \rangle - \frac{(-1)^\beta (2\alpha - 2\beta + 3) + (-1)^\alpha}{2\alpha + 3} \langle c_1^{2\alpha+1} c_2 \mathfrak{b}_{m-1} e_m \rangle$$
$$= -\frac{c_2^{m+\beta+2}}{c_1} \mathfrak{b}_{\alpha-\beta-1} \mathfrak{b}_{\alpha-\beta}$$

and

(3.15)
$$\left[c_1^{m-3-2\alpha-2\beta} c_2^{2\alpha+\beta+1} \right] - \frac{(-1)^\beta (m-1-2\alpha+2\beta) + (-1)^{\frac{m}{2}-\alpha}}{m-1-2\alpha} \left[c_1^{m-3-2\alpha} c_2^{2\alpha+1} \mathfrak{b}_{m-1} e_m \right]$$
$$= -\frac{c_2^{m+2\alpha+\beta+1}}{c_1} \mathfrak{b}_\gamma \mathfrak{b}_{\gamma+1}, \quad \text{where } \gamma = \frac{m}{2} - 2 - \alpha - \beta.$$

Last we consider a generator $c_1^{2i+2} c_2^j \mathfrak{b}_{m-1} e_m$. We set

$$\langle c_1^{2i+2} c_2^j \mathfrak{b}_{m-1} e_m \rangle := c_1^{2i+2} c_2^j \mathfrak{b}_{m-1} e_m - \sum_{\mu=1}^{i} a_\mu c_2^{j+\mu} c_1^{2i+2-2\mu} \mathfrak{b}_{m-1} e_m$$
$$- \left((-1)^i (2i+3) + 1 \right) c_2^{i+j+1} \mathfrak{b}_{m-1} e_m,$$

where $f_{2i+3}(x) = \sum_{\mu=1}^{i+1} a_\mu x^\mu (1+x)^{2i+3-2\mu}$ (see Lemma 3.5). Then the formula (3.9) implies that

$$(3.16) \qquad \langle c_1^{2i+1} c_2^j \mathfrak{b}_{m-1} e_m \rangle = -c_2^{m+j} \mathfrak{b}_i \mathfrak{b}_{i+1}.$$

We consider a set

$$\{(3.6), (3.10), (3.11), (3.12), (3.13), (3.14)\}$$

$$\cup \{x \in \mathcal{A}_n : x = c_1^i c_2^j e_m^{\epsilon_1}, x = c_1^i \mathfrak{b}_{m-1}, i+j \leq m-2, \epsilon_k = 0 \text{ or } 1\}.$$

When we reduce it to the mod p reduction for an odd prime p, the identities from (3.6) to (3.14) show that they are linearly independent from Lemma 3.6. Hence, replacing $\mathfrak{b}_{m-1}$ and e_m by v_{2m-2} and v_{2m} respectively, we obtain an integral basis of $CH^\cdot(Y_{2m+1})$ for even m. An integral basis of $CH^\cdot(Y_n)$ for other cases is written down in a table of Sect. 5.

For a group G and a subgroup H, let $[G : H]$ denote the index, i.e. the cardinality of G/H. As a corollary of the above theorem, we have

Corollary 3.8. (i) *For $n = 2m$,*

$$[CH^\cdot(Y_n) : A_n] = \begin{cases} 1^2 \cdot 3^2 \cdot \ldots \cdot (m-3)^2 \cdot (m-1) & m: \text{ even} \\ 1^2 \cdot 3^2 \cdot \ldots \cdot (m-2)^2 & m: \text{ odd}. \end{cases}$$

(ii) *For $n = 2m+1$,*

$$[CH^\cdot(Y_n) : A_n] = \begin{cases} 1^2 \cdot 3^2 \cdot \ldots \cdot (m-3)^2 \cdot (m-1) & m: \text{ even} \\ 1^2 \cdot 3^2 \cdot \ldots \cdot (m-2)^2 \cdot m & m: \text{ odd}. \end{cases}$$

The following theorem is proved by using the integral basis of $CH^\cdot(Y_n)$ given in Theorem 3.7.

Theorem 3.9. *The ring structure of $CH^\cdot(Y_n)$ is determined. The results are summed up in tables in 5.3 and 5.4 in Sect. 5. (See also 5.5.)*

Proof . We only show the formula (iii) in Sect. 5, 5.4. We use the notations of the proof of Theorem 3.7. Using $x^{2i} \mathfrak{d}_{m-1-i} = 0$, we have

$$c_1^{m-2i-1} c_2^{2i} \mathfrak{b}_{m-1} - x^{2i} \mathfrak{d}_{m-1-i} = \frac{x^m}{x^2-1}(x^{2i}-1) + \frac{x^{2i}}{x^2-1}(x^m-1)f_{m-2i}(x).$$

We set $I_1 = \frac{x^m}{x^2-1}(x^{2i}-1)$ and $I_2 = \frac{x^{2i}}{x^2-1}(x^m-1)f_{m-2i}(x)$. Then we have

$$I_2 = \sum_{\mu=1}^{\frac{m}{2}-i-1} a_\mu c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \mathfrak{b}_{m-1} + (-1)^{\frac{m}{2}-i+1} 2x^{\frac{m}{2}+i} \frac{x^m-1}{x^2-1}.$$

We set $J = x^{\frac{m}{2}+i}\frac{x^m-1}{x^2-1}$. From (3.6), we have

$$J - c_2^{\frac{m}{2}+i-1}e_m - \left(\sum_{\mu=0}^{\frac{m}{2}-2-i}(-1)^\mu\left[c_1^{1+2\mu}c_2^{\frac{m}{2}+i-1-\mu}\mathfrak{b}_{m-1}\right]\right) = (-1)^{\frac{m}{2}-i}x^m\frac{x^{2i}-1}{x^2-1}.$$

Hence

(3.17)

$$c_1^{m-2i-1}c_2^{2i}\mathfrak{b}_{m-1} - \sum_{\mu=1}^{\frac{m}{2}-i-1}a_\mu c_1^{m-2i-1-2\mu}c_2^{2i+\mu}\mathfrak{b}_{m-1}$$

$$+ (-1)^{\frac{m}{2}-i}2\left(c_2^{\frac{m}{2}+i-1}e_m + \sum_{\mu=0}^{\frac{m}{2}-2-i}(-1)^\mu\left[c_1^{2\mu+1}c_2^{\frac{m}{2}+i-1-\mu}\mathfrak{b}_{m-1}\right]\right) = -x^m\frac{x^{2i}-1}{x^2-1}.$$

On the other hand, we see from (3.13) that

(3.18) $\qquad -x^m\frac{x^{2i}-1}{x^2-1} = \langle c_1^{2i-1}\mathfrak{b}_{m-1}e_m\rangle + \frac{2i+1}{2i-1}\langle c_1^{2i-3}c_2\mathfrak{b}_{m-1}e_m\rangle.$

We calculate

$$S := (-1)^{\frac{m}{2}-i}2\left(c_2^{\frac{m}{2}+i-1}e_m + \sum_{\mu=0}^{\frac{m}{2}-2-i}(-1)^\mu\left[c_1^{2\mu+1}c_2^{\frac{m}{2}+i-1-\mu}\mathfrak{b}_{m-1}\right]\right).$$

By (3.6),

$$S = (-1)^{\frac{m}{2}-i}2(S_1 + S_2),$$

where

$$S_1 = \sum_{\mu=0}^{\frac{m}{2}-2-i}(-1)^\mu c_2^{\frac{m}{2}+i+1-\mu}\mathfrak{b}_{2\mu+1}\mathfrak{b}_{m-1} \quad\text{and}\quad S_2 = c_2^{\frac{m}{2}+i-1}e_m - \sum_{\mu=0}^{\frac{m}{2}-2-i}(-1)^\mu c_2^{\frac{m}{2}+i-2-\mu}\mathfrak{b}_{2\mu+2}e_m.$$

We have from Lemma 3.3 that

$$S_1 = \sum_{\mu=1}^{\frac{m}{2}-1-i}(-1)^{\frac{m}{2}-1-\mu-i}\binom{m-1-2i-\mu}{\mu-1}c_1^{m-2i-1-2\mu}c_2^{2i+\mu}\mathfrak{b}_{m-1}.$$

Using $\sum_{\mu=0}^{h}(-1)^\mu c_2^{h-\mu}\mathfrak{b}_{2\mu} = (-1)^h\mathfrak{d}_h$ (see Lemma 3.4),

$$S_2 = c_2^{2i}\left(\sum_{\mu=0}^{\frac{m}{2}-1-i}(-1)^\mu c_2^{\frac{m}{2}-i-1-\mu}\mathfrak{b}_{2\mu}\right)\mathfrak{b}_{m-1} = (-1)^{\frac{m}{2}-1-i}c_2^{2i}\mathfrak{d}_{\frac{m}{2}-1-i}\mathfrak{b}_{m-1}.$$

The formula (3.17) is

$$c_1^{m-2i-1}c_2^{2i}\mathfrak{b}_{m-1} + \sum_{\mu=1}^{\frac{m}{2}-i-1} (-1)^\mu \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1}c_2^{2i+\mu}\mathfrak{b}_{m-1} - 2c_2^{2i}\mathfrak{d}_{\frac{m}{2}-1-i}e_m$$

$$= -x^m \frac{x^{2i-1}}{x^2-1}.$$

Comparing with (3.18), we obtain

$$c_1^{m-2i-1}c_2^{2i}\mathfrak{b}_{m-1} = \sum_{\mu=1}^{\frac{m}{2}-i-1} (-1)^{\mu+1} \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1}c_2^{2i+\mu}\mathfrak{b}_{m-1}$$

$$- \frac{2}{2i-1}c_2^{2i}\mathfrak{d}_{\frac{m}{2}-1-i}e_m + \left[c_1^{2i-1}\mathfrak{b}_{m-1}e_m\right] + \frac{2i+1}{2i-1}\left[c_1^{2i-3}c_2\mathfrak{b}_{m-1}e_m\right].$$

This completes the proof of the formula. $\square$

4. The Chow ring of X_n

Theorem 4.1. *Let $T(X_n)$ and $F(X_n)$ be the torsion part and the free part of $CH^\cdot(X_n)$, respectively. Then we have*

(i) *For $n = 4t$,*

$$F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t)\{1, v_{4t-4}\}$$
$$T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t)\{v_{4t-2}, v_{4t-4}v_{4t-2}\}.$$

(ii) *For $n = 4t + 1$,*

$$F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t) \oplus \mathbb{Z}[c_2]/(c_2^{t-1})\{v_{4t}\}$$
$$T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t)\{v_{4t-2}, v_{4t-2}v_{4t}\} \oplus \mathbb{Z}/(2t)\{c_2^{t-1}v_{4t}\}.$$

(iii) *For $n = 4t + 2$,*

$$F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t)\{1, v_{4t}\} \oplus \mathbb{Z}\{v_{4t-2}\}$$
$$T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^{t-1})\{c_2v_{4t-2}, c_2v_{4t-2}v_{4t}\} \oplus \mathbb{Z}/4\{v_{4t-2}v_{4t}\}.$$

(iv) *For $n = 4t + 3$,*

$$F(X_n) \simeq \mathbb{Z}[c_2]/(c_2^t)\{1, v_{4t}\}$$
$$T(X_n) \simeq \mathbb{Z}/2[c_2]/(c_2^t)\{v_{4t+2}, v_{4t}v_{4t+2}\} \oplus \mathbb{Z}/(2t+1)\{c_2^tv_{4t}\}.$$

Proof . Let $\tilde{X}_n = X_n \times_{\mathbb{G}_m} \mathbb{A}^1$ be the associated bundle of (3.1) and $s : Y_n \to \tilde{X}_n$ the 0-section. Since $s^* : CH^\cdot(\tilde{X}_n) \xrightarrow{\sim} CH^\cdot(Y_n)$ by Theorem 2.4 (ii), the first assertion of the same theorem for $V = \tilde{X}_n$ and $X = s(Y_n)$ gives an exact sequence

$$CH^\cdot(Y_n) \xrightarrow{\cdot c_1} CH^\cdot(Y_n) \xrightarrow{\pi^*} CH^\cdot(X_n) \to 0.$$

Theorem 4.1 follows from this and the ring structure of $CH^{\cdot}(Y_n)$ in Theorems 3.7 and 3.9. $\qquad\square$

Next we consider the cycle map. The cohomology groups mean an etale cohomology [9], [13]. All varieties are defined over K', which is a subfield of an algebraically closed field K. Let l be a prime with $(l, \mathrm{ch}(K)) = 1$. We denote a locally constant sheaf $\mu_l^{\otimes i}$ by $\mathbb{Z}/l^{(i)}$.

Corollary 4.2. *The homomorphism* $cl : CH^i(X_n) \to H^{2i}(X_n, \mathbb{Z}/l^{(i)})$ *is injective.*

Proof . Since $(\tilde{X}_n, Y_n)$ is a smooth pair, we have the Gysin sequence as in [5, Appendice 1.3.3] and [13, VI Remark 5.4]. Since the cycle map and the Gysin map are commutative, we have the following commutative diagram, where each row is exact:

$$
\begin{array}{ccccccc}
CH^i(Y_n) & \xrightarrow{\cdot c_1} & CH^{i+1}(Y_n) & \xrightarrow{\pi^*} & CH^{i+1}(X_n) & \longrightarrow & 0 \\
\downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle cl} & & \downarrow{\scriptstyle cl} & & \\
H^{2i}(Y_n, \mathbb{Z}/l^{(i)}) & \xrightarrow{\cdot c_1} & H^{2(i+1)}(Y_n, \mathbb{Z}/l^{(i+1)}) & \xrightarrow{\pi^*} & H^{2(i+1)}(X_n, \mathbb{Z}/l^{(i+1)}) & &
\end{array}
$$

Corollary 4.2 follows from Theorem 4.1 and this diagram. $\qquad\square$

Remark 4.3. Assume that we have a K'-isomorphism $Y_n \simeq SO(n, K)/(SO(n-4, K) \times GL(2, K))$, where $SO(n, K)$ and $SO(n - 4, K)$ are split over K', and that $\pi : Y_n \to X_n$ is a K'-map. Then the Galois actions $\overline{G} = \mathrm{Gal}(\overline{K}'/K')$ on $H^{\cdot}(Y_n, \mathbb{Z}/l^{(i)})$ and $H^{\cdot}(X_n, \mathbb{Z}/l^{(i)})$ are described by the character group $X(T)$ of a K'-split maximal torus T of $SO(n, K)$. It follows from a result of [5, 8-2].

5. TABLES OF THE RING STRUCTURE OF $CH^{\cdot}(Y_n)$

5.1. Notations. (i) For $k \in \mathbb{N} \cup \{0\}$, we define b_k and $d_k \in \mathbb{Z}[c_1, c_2]$ as follows:

$$
b_k = (-1)^k \sum_{\mu=0}^{\left[\frac{k}{2}\right]} (-1)^\mu \binom{k - \mu}{\mu} c_1^{k-2\mu} c_2^{\mu}
$$

and

$$
d_k = (-1)^k \sum_{\mu=0}^{k} (-1)^\mu \binom{2k - \mu + 1}{\mu} c_1^{2k-2\mu} c_2^{\mu}.
$$

(ii) For $g \in \mathbb{N}$ and $\mu \in \mathbb{N} \cup \{0, -1\}$, we define $a_{g,\mu} \in \mathbb{Z}$ by

$$
a_{g,\mu} = \begin{cases} (-1)^{1+\mu} \frac{g}{\mu} \binom{g-1-\mu}{\mu-1} & \mu \geq 1 \\ -1 & \mu = 0 \\ 0 & \mu = -1. \end{cases}
$$

Then the integers $a_{g,\mu}$ are characterized by

$$
(1+x)^g = 1 + x^g + \sum_{\mu=1}^{\left[\frac{g}{2}\right]} a_{g,\mu} x^\mu (1+x)^{g-2\mu}.
$$

(iii) The generators v_{2i} are given by (3.2) and (3.3).

5.2. An integral basis of $CH^\cdot(Y_n)$. In the following (I) and (II), we give an integral basis of $CH^\cdot(Y_n)$. The notations are explained as follows: Let S_n be the set of the monomial basis of A_n in (3.4). Let T be a subset of S_n. Then for an element $\xi \in T$, $\langle \xi \rangle$ (resp. $\langle \xi \rangle'$) is defined to be the right-hand side of an equation (1)-(8) below. We consider a set

$$
\left\{ \frac{\langle \xi \rangle}{l_\xi} : \xi \in T \right\} \cup \{\eta : \eta \in S_n - T\},
$$

where $l_\xi \in \mathbb{N}$. Following this procedure, we obtain an integral basis of $CH^\cdot(Y_n)$. We abbreviate this basis as $\left\{ \dfrac{\langle \xi \rangle}{l_\xi} : \xi \in T \right\}$.

(I) The case $n = 2m$.
    (i) For even m,

$$
\left\{ \frac{\langle c_1^{2i+1} c_2 v_{2m-4} v_{2m-2} \rangle}{2i+3}, \frac{\langle c_1^{m-2j-3} c_2^{2j+1} v_{2m-4} v_{2m-2} \rangle}{m-2j-1} : 0 \leq i \leq \frac{m}{2} - 2, 1 \leq j \leq \frac{m}{2} - 2 \right\}.
$$

(ii) For odd m,

$$
\left\{ \frac{\langle c_1^{2i+1} c_2 v_{2m-4} v_{2m-2} \rangle'}{2i+3}, \frac{\langle c_1^{m-2j-2} c_2^{2j} v_{2m-4} v_{2m-2} \rangle'}{m-2j} : 0 \leq i \leq \frac{m-5}{2}, 1 \leq j \leq \frac{m-3}{2} \right\}.
$$

(II) The case $n = 2m+1$.
    (iii) For even m,

$$
\left\{ \frac{\langle c_1^{2i+1} c_2 v_{2m-2} v_{2m} \rangle}{2i+3}, \frac{\langle c_1^{m-2j-3} c_2^{2j+1} v_{2m-2} v_{2m} \rangle}{m-2j-1} : 0 \leq i \leq \frac{m}{2} - 2, 1 \leq j \leq \frac{m}{2} - 2 \right\}.
$$

(iv) For odd m,

$$\left\{\frac{\langle c_1^{2i+1}v_{2m-2}v_{2m}\rangle}{2i+3}, \frac{\langle c_1^{m-2j-2}c_2^{2j}v_{2m-2}v_{2m}\rangle}{m-2j} : 0 \leq i \leq \frac{m-3}{2}, 1 \leq j \leq \frac{m-3}{2}\right\}.$$

Here $\langle \ \ \rangle$ and $\langle \ \ \rangle'$ are defined as follows:

(1)

$$\langle c_1^{2i+1}c_2 v_{2m-4}v_{2m-2}\rangle = c_1^{2i+1}c_2 v_{2m-4}v_{2m-2} + (-1)^{\frac{m+2i+2}{2}}\frac{(-1)^i(2i+3)+1}{2}c_2^{2i+4}d_{\frac{m-2i-6}{2}}v_{2m-4}$$

$$-\sum_{\mu=1}^{i}a_{2i+3,\mu}c_1^{2i+1-2\mu}c_2^{1+\mu}v_{2m-4}v_{2m-2}.$$

(2)
$$\langle c_1^{m-2j-3}c_2^{2j+1}v_{2m-4}v_{2m-2}\rangle = c_1^{m-2j-3}c_2^{2j+1}v_{2m-4}v_{2m-2}$$

$$-\sum_{\mu=1}^{\frac{m-2j-4}{2}}a_{m-2j-1,\mu}c_1^{m-2j-3-2\mu}c_2^{2j+1+\mu}v_{2m-4}v_{2m-2}.$$

(3)
$$\langle c_1^{2i+1}c_2 v_{2m-4}v_{2m-2}\rangle' = c_1^{2i+1}c_2 v_{2m-4}v_{2m-2} + (-1)^{\frac{m+2i+1}{2}}\frac{(-1)^i(2i+3)+1}{2}c_2^{2i+3}d_{\frac{m-2i-5}{2}}v_{2m-2}$$

$$-\sum_{\mu=1}^{i}a_{2i+3,\mu}c_1^{2i+1-2\mu}c_2^{1+\mu}v_{2m-4}v_{2m-2}.$$

(4)
$$\langle c_1^{m-2j-2}c_2^{2j}v_{2m-4}v_{2m-2}\rangle' = c_1^{m-2j-2}c_2^{2j}v_{2m-4}v_{2m-2}$$

$$-\sum_{\mu=1}^{\frac{m-2j-3}{2}}a_{m-2j,\mu}c_1^{m-2j-2-2\mu}c_2^{2j+\mu}v_{2m-4}v_{2m-2}.$$

(5)
$$\langle c_1^{2i+1}c_2 v_{2m-2}v_{2m}\rangle = c_1^{2i+1}c_2 v_{2m-2}v_{2m} + (-1)^{\frac{m+2i+2}{2}}\frac{(-1)^i(2i+3)+1}{2}c_2^{2i+4}d_{\frac{m-2i-6}{2}}v_{2m}$$

$$-\sum_{\mu=1}^{i}a_{2i+3,\mu}c_1^{2i+1-2\mu}c_2^{1+\mu}v_{2m-2}v_{2m}.$$

(6)
$$\langle c_1^{m-2j-3}c_2^{2j+1}v_{2m-2}v_{2m}\rangle = c_1^{m-2j-3}c_2^{2j+1}v_{2m-2}v_{2m}$$

$$- \sum_{\mu=1}^{\frac{m-2j-4}{2}} a_{m-2j-1,\mu} c_1^{m-2j-3-2\mu} c_2^{2j+1+\mu} v_{2m-2} v_{2m}.$$

(7)

$$\langle c_1^{2i+1} v_{2m-2} v_{2m} \rangle = c_1^{2i+1} v_{2m-2} v_{2m} + (-1)^{\frac{m+2i+3}{2}} \frac{(-1)^i (2i+3)+1}{2} c_2^{2i+3} d_{\frac{m-2i-5}{2}} v_{2m-2}$$

$$- \sum_{\mu=1}^{i} a_{2i+3,\mu} c_1^{2i+1-2\mu} c_2^{\mu} v_{2m-2} v_{2m}.$$

(8)

$$\langle c_1^{m-2j-2} c_2^{2j} v_{2m-2} v_{2m} \rangle = c_1^{m-2j-2} c_2^{2j} v_{2m-2} v_{2m}$$

$$- \sum_{\mu=1}^{\frac{m-2j-3}{2}} a_{m-2j,\mu} c_1^{m-2j-2-2\mu} c_2^{2j+\mu} v_{2m-2} v_{2m}.$$

5.3. The ring structure of $CH^{\cdot}(Y_n)_{(2)}$ for $n = 2m$.

	even m	odd m
c_1^{m-1}	(1)	
$c_1^{m-k-1} c_2^{k} \, (k \geq 1)$	(2)	
$c_1^{m-1} v_{2m-4}$	(3)	
$c_1^{m-2i-1} c_2^{2i} v_{2m-4} \, (i \geq 1)$	(4)	
$c_1^{m-2i-2} c_2^{2i+1} v_{2m-4} \, (i \geq 0)$	(5)	(6)
$c_1^{m-2i-1} c_2^{2i} v_{2m-2} \, (i \geq 0)$	(7)	(8)
$c_1^{m-2i-2} c_2^{2i+1} v_{2m-2} \, (i \geq 0)$	(9)	
$c_1^{m-2i-1} c_2^{2i} v_{2m-4} v_{2m-2} \, (i \geq 0)$	(10)	(11)
$c_1^{m-2i-2} c_2^{2i+1} v_{2m-4} v_{2m-2} \, (i \geq 0)$	(12)	(13)
v_{2m-4}^2	(14)	(15)
v_{2m-2}^2	(16)	(17)

Here

$$(1) = \left\{ \sum_{\mu=1}^{\left[\frac{m-1}{2}\right]} (-1)^{1+\mu} \binom{m-1-\mu}{\mu} c_1^{m-1-2\mu} c_2^{\mu} \right\} + (-1)^{m+1} 2 v_{2m-2}.$$

$$(2) = \left\{ \sum_{\mu=1}^{\left[\frac{m-k-1}{2}\right]} (-1)^{1+\mu} \binom{m-k-1-\mu}{\mu} c_1^{m-k-1-2\mu} c_2^{k+\mu} \right\}$$

$$+ \left\{ (-1)^{m+k} 2c_2 b_{k-1} \right\} v_{2m-4} + \left\{ (-1)^{m+k} 2c_2 b_{k-2} \right\} v_{2m-2}.$$

$$(3) = \left\{ \sum_{\mu=1}^{\left[\frac{m-1}{2}\right]} (-1)^{1+\mu} \binom{m-1-\mu}{\mu} c_1^{m-1-2\mu} c_2^{\mu} \right\} v_{2m-4} + (-1)^{m+1} 2 v_{2m-4} v_{2m-2}.$$

$$(4) = \left\{ \sum_{\mu=1}^{\left[\frac{m-2i-1}{2}\right]} (-1)^{1+\mu} \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-4}$$

$$+ \left\{ (-1)^m 2 \sum_{\mu=0}^{i-1} a_{2i-1,\mu} c_1^{2i-2-2\mu} c_2^{1+\mu} \right\} v_{2m-4} v_{2m-2}.$$

$$(5) = \left\{ (-1)^{\frac{m+2i+2}{2}} \frac{4i}{2i+1} c_2^{2i+2} d_{\frac{m-2i-4}{2}} + \sum_{\mu=1}^{\frac{m-2i-2}{2}} a_{m-2i-1,\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-4}$$

$$+ \left\{ 2 \sum_{\mu=-1}^{i-2} \left(a_{2i-1,\mu} + \frac{2i-1}{2i+1} a_{2i+1,1+\mu} \right) c_1^{2i-3-2\mu} c_2^{2+\mu} \right\} v_{2m-4} v_{2m-2}.$$

$$(6) = \left\{ \sum_{\mu=1}^{\frac{m-2i-3}{2}} (-1)^{1+\mu} \binom{m-2i-2-\mu}{\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-4}$$

$$+ \left\{ (-1)^{\frac{m+2i+1}{2}} \frac{2}{2i+1} c_2^{2i+1} d_{\frac{m-2i-3}{2}} \right\} v_{2m-2}$$

$$- \left\{ 2 \sum_{\mu=-1}^{i-2} \left(a_{2i-1,\mu} + \frac{2i-1}{2i+1} a_{2i+1,1+\mu} \right) c_1^{2i-3-2\mu} c_2^{2+\mu} \right\} v_{2m-4} v_{2m-2}.$$

$$(7) = \left\{ (-1)^{\frac{m+2i}{2}} \frac{2}{2i+1} c_2^{2i+2} d_{\frac{m-2i-4}{2}} \right\} v_{2m-4}$$

$$+ \left\{ \sum_{\mu=1}^{\frac{m-2i-2}{2}} (-1)^{1+\mu} \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-2}$$

$$+ \left\{ 2 \sum_{\mu=-1}^{i-2} \left(a_{2i-1,\mu} + \frac{2i-1}{2i+1} a_{2i+1,1+\mu} \right) c_1^{2i-3-2\mu} c_2^{2+\mu} \right\} v_{2m-4} v_{2m-2}.$$

$$(8) = \left\{ (-1)^{\frac{m+2i+3}{2}} \frac{4i}{2i+1} c_2^{2i+1} d_{\frac{m-2i-3}{2}} + \sum_{\mu=1}^{\frac{m-2i-1}{2}} a_{m-2i,\mu} c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-2}$$

$$-\left\{ 2\sum_{\mu=-1}^{i-2}\left(a_{2i-1,\mu}+\frac{2i-1}{2i+1}a_{2i+1,1+\mu}\right)c_1^{2i-3-2\mu}c_2^{2+\mu}\right\}v_{2m-4}v_{2m-2}.$$

$$(9)=\left\{\sum_{\mu=1}^{\left[\frac{m-2i-2}{2}\right]}(-1)^{1+\mu}\binom{m-2i-2-\mu}{\mu}c_1^{m-2i-2-2\mu}c_2^{2i+1+\mu}\right\}v_{2m-2}$$

$$+\left\{(-1)^m2\sum_{\mu=0}^{i}a_{2i+1,\mu}c_1^{2i-2\mu}c_2^{1+\mu}\right\}v_{2m-4}v_{2m-2}.$$

$$(10)=\left\{\sum_{\mu=0}^{\frac{m-2i-4}{2}}\left(\frac{m-2i+1}{m-2i-1}a_{m-2i-1,\mu}+a_{m-2i+1,1+\mu}\right)c_1^{m-2i-3-2\mu}c_2^{2i+1+\mu}\right\}v_{2m-4}v_{2m-2}.$$

$$(11)=\left\{\sum_{\mu=1}^{\frac{m-2i-1}{2}}a_{m-2i,\mu}c_1^{m-2i-1-2\mu}c_2^{2i+\mu}\right\}v_{2m-4}v_{2m-2}.$$

$$(12)=\left\{\sum_{\mu=1}^{\frac{m-2i-2}{2}}a_{m-2i-1,\mu}c_1^{m-2i-2-2\mu}c_2^{2i+1+\mu}\right\}v_{2m-4}v_{2m-2}.$$

$$(13)=\left\{\sum_{\mu=0}^{\frac{m-2i-5}{2}}\left(\frac{m-2i}{m-2i-2}a_{m-2i-2,\mu}+a_{m-2i,1+\mu}\right)c_1^{m-2i-4-2\mu}c_2^{2i+2+\mu}\right\}v_{2m-4}v_{2m-2}.$$

$$(14)=(-1)^{\frac{m}{2}}d_{\frac{m-2}{2}}v_{2m-4}.$$

$$(15)=-b_{m-2}v_{2m-4}+(-1)^{\frac{m+3}{2}}d_{\frac{m-3}{2}}v_{2m-2}.$$

$$(16)=(-1)^{\frac{m+2}{2}}c_2^2d_{\frac{m-4}{2}}v_{2m-4}.$$

$$(17)=(-1)^{\frac{m+1}{2}}c_2d_{\frac{m-3}{2}}v_{2m-2}.$$

5.4. The ring structure of $CH^{\cdot}(Y_n)_{(2)}$ for $n = 2m + 1$.

	even m	odd m
c_1^{m-1}	(i)	
$c_1^{m-k-1}c_2^k\ (k \geq 1)$	(ii)	
$c_1^{m-2i-1}c_2^{2i}v_{2m-2}\ (i \geq 0)$	(iii)	(iv)
$c_1^{m-2i-2}c_2^{2i+1}v_{2m-2}\ (i \geq 0)$	(v)	
$c_1^{m-1}v_{2m}$	(vi)	
$c_1^{m-2i-1}c_2^{2i}v_{2m}\ (i \geq 1)$	(vii)	
$c_1^{m-2i-2}c_2^{2i+1}v_{2m}\ (i \geq 0)$	(viii)	(ix)
$c_1^{m-2i-1}c_2^{2i}v_{2m-2}v_{2m}\ (i \geq 0)$	(x)	(xi)
$c_1^{m-2i-2}c_2^{2i+1}v_{2m-2}v_{2m}\ (i \geq 0)$	(xii)	(xiii)
v_{2m-2}^2	(xiv)	(xv)
v_{2m}^2	(xvi)	(xvii)

Here

$$(\text{i}) = \left\{ \sum_{\mu=1}^{[\frac{m-1}{2}]} (-1)^{1+\mu} \binom{m-1-\mu}{\mu} c_1^{m-1-2\mu} c_2^{\mu} \right\} + (-1)^{m+1} 2 v_{2m-2}.$$

$$(\text{ii}) = \left\{ \sum_{\mu=1}^{[\frac{m-k-1}{2}]} (-1)^{1+\mu} \binom{m-k-1-\mu}{\mu} c_1^{m-k-1-2\mu} c_2^{k+\mu} \right\}$$
$$+ \left\{ (-1)^{m+k} 2 c_2 b_{k-2} \right\} v_{2m-2} + \left\{ (-1)^{m+k+1} 2 b_{k-1} \right\} v_{2m}.$$

$$(\text{iii}) = \left\{ \sum_{\mu=1}^{\frac{m-2i-2}{2}} (-1)^{1+\mu} \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-2}$$
$$+ \left\{ (-1)^{\frac{m+2i+2}{2}} \frac{2}{2i-1} c_2^{2i} d_{\frac{m-2i-2}{2}} \right\} v_{2m}$$
$$- \left\{ 2 \sum_{\mu=0}^{i-1} \left(\frac{2i+1}{2i-1} a_{2i-1,-1+\mu} + a_{2i+1,\mu} \right) c_1^{2i-1-2\mu} c_2^{\mu} \right\} v_{2m-2} v_{2m}.$$

$$(\text{iv}) = \left\{ (-1)^{\frac{m+2i+3}{2}} \frac{4i}{2i+1} c_2^{2i+1} d_{\frac{m-2i-3}{2}} + \sum_{\mu=1}^{\frac{m-2i-1}{2}} a_{m-2i,\mu} c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-2}$$
$$+ \left\{ 2 \sum_{\mu=-1}^{i-2} \left(a_{2i-1,\mu} + \frac{2i-1}{2i+1} a_{2i+1,1+\mu} \right) c_1^{2i-3-2\mu} c_2^{1+\mu} \right\} v_{2m-2} v_{2m}.$$

$$\text{(v)} = \left\{ \sum_{\mu=1}^{\left[\frac{m-2i-2}{2}\right]} (-1)^{1+\mu} \binom{m-2i-2-\mu}{\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2}$$

$$+ \left\{ (-1)^{m+1} 2 \sum_{\mu=0}^{i} a_{2i+1,\mu} c_1^{2i-2\mu} c_2^{\mu} \right\} v_{2m-2} v_{2m}.$$

$$\text{(vi)} = \left\{ \sum_{\mu=1}^{\left[\frac{m-1}{2}\right]} (-1)^{1+\mu} \binom{m-1-\mu}{\mu} c_1^{m-1-2\mu} c_2^{\mu} \right\} v_{2m} + (-1)^{m+1} 2 v_{2m-2} v_{2m}.$$

$$\text{(vii)} = \left\{ \sum_{\mu=1}^{\left[\frac{m-2i-1}{2}\right]} (-1)^{1+\mu} \binom{m-2i-1-\mu}{\mu} c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m}$$

$$+ \left\{ (-1)^{m} 2 \sum_{\mu=0}^{i-1} a_{2i-1,\mu} c_1^{2i-2-2\mu} c_2^{1+\mu} \right\} v_{2m-2} v_{2m}.$$

$$\text{(viii)} = \left\{ (-1)^{\frac{m+2i+2}{2}} \frac{4i}{2i+1} c_2^{2i+2} d_{\frac{m-2i-4}{2}} + \sum_{\mu=1}^{\frac{m-2i-2}{2}} a_{m-2i-1,\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m}$$

$$+ \left\{ 2 \sum_{\mu=-1}^{i-2} \left(a_{2i-1,\mu} + \frac{2i-1}{2i+1} a_{2i+1,1+\mu} \right) c_1^{2i-3-2\mu} c_2^{2+\mu} \right\} v_{2m-2} v_{2m}.$$

$$\text{(ix)} = \left\{ (-1)^{\frac{m+2i+3}{2}} \frac{2}{2i+3} c_2^{2i+3} d_{\frac{m-2i-5}{2}} \right\} v_{2m-2}$$

$$+ \left\{ \sum_{\mu=1}^{\frac{m-2i-3}{2}} (-1)^{1+\mu} \binom{m-2i-2-\mu}{\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m}$$

$$+ \left\{ 2 \sum_{\mu=-1}^{i-1} \left(a_{2i+1,\mu} - a_{2i+1,1+\mu} + \frac{2i+1}{2i+3} a_{2i+3,1+\mu} \right) c_1^{2i-1-2\mu} c_2^{1+\mu} \right\} v_{2m-2} v_{2m}.$$

$$\text{(x)} = \left\{ \sum_{\mu=0}^{\frac{m-2i-4}{2}} \left(\frac{m-2i+1}{m-2i-1} a_{m-2i-1,\mu} + a_{m-2i+1,1+\mu} \right) c_1^{m-2i-3-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} v_{2m}.$$

$$\text{(xi)} = \left\{ \sum_{\mu=1}^{\frac{m-2i-1}{2}} a_{m-2i,\mu} c_1^{m-2i-1-2\mu} c_2^{2i+\mu} \right\} v_{2m-2} v_{2m}.$$

$$(\text{xii}) = \left\{ \sum_{\mu=1}^{\frac{m-2i-2}{2}} a_{m-2i-1,\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \right\} v_{2m-2} v_{2m}.$$

$$(\text{xiii}) = \left\{ \sum_{\mu=0}^{\frac{m-2i-5}{2}} \left(\frac{m-2i}{m-2i-2} a_{m-2i-2,\mu} + a_{m-2i,1+\mu} \right) c_1^{m-2i-4-2\mu} c_2^{2i+2+\mu} \right\} v_{2m-2} v_{2m}.$$

$$(\text{xiv}) = (-1)^{\frac{m+2}{2}} d_{\frac{m-2}{2}} v_{2m}.$$

$$(\text{xv}) = (-1)^{\frac{m+1}{2}} c_2 d_{\frac{m-3}{2}} v_{2m-2}.$$

$$(\text{xvi}) = (-1)^{\frac{m}{2}} c_2^2 d_{\frac{m-4}{2}} v_{2m}.$$

$$(\text{xvii}) = (-1)^{\frac{m+1}{2}} \frac{1}{3} c_2^3 d_{\frac{m-5}{2}} v_{2m-2} - \frac{2}{3} c_1 v_{2m-2} v_{2m}.$$

5.5. A remark on the ring structure of $CH^{\cdot}(Y_n)$. We have given the ring structure of $CH^{\cdot}(Y_n)_{(2)}$ in 5.3 and 5.4. But actually, it is easy to determine the ring structure of $CH^{\cdot}(Y_n)$ from 5.2, 5.3 and 5.4. For example, by the basis in 5.2, the formula 5.3 (5) is rewritten as follows:

$$(5)' \quad c_1^{m-2i-2} c_2^{2i+1} v_{2m-4} = \left\{ (-1)^{\frac{m+2i+2}{2}} \left((-1)^i (2i-1) + 1 \right) c_2^{2i+2} d_{\frac{m-2i-4}{2}} \right.$$

$$+ \sum_{\mu=1}^{\frac{m-2i-2}{2}} a_{m-2i-1,\mu} c_1^{m-2i-2-2\mu} c_2^{2i+1+\mu} \Bigg\} v_{2m-4}$$

$$+ \left\{ 2 \sum_{\mu=-1}^{i-2} a_{2i-1,\mu} c_1^{2i-3-2\mu} c_2^{2+\mu} \right\} v_{2m-4} v_{2m-2} - \frac{4i-2}{2i+1} \langle c_1^{2i-1} c_2 v_{2m-4} v_{2m-2} \rangle.$$

The other cases can be calculated similarly.

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